ON THE GENERALIZED INPUT ESTIMATION

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1. Introduction

Consider the problem of maneuvering target tracking within the framework of the familiar time invariant linear dynamic system [1]

$$x_{k+1} = Fx_k + Gu_k + w_k \tag{1}$$

$$z_{k+1} = Hx_{k+1} + v_{k+1}, \qquad k = 0, 1, 2, \ldots \tag{2}$$

where $x_k \in \mathbf{R}^{n_x}$ denotes the target state with transition matrix F, $u_k \in \mathbf{R}^{n_u}$ is the control input with transition matrix G, $z_k \in \mathbf{R}^{n_z}$ is the measurement with measurement matrix H, and $w_k \in \mathbf{R}^{n_x}$, $v_k \in \mathbf{R}^{n_z}$ denote respectively the process noise and measurement errors which are assumed independent Gaussian white noises with zero means and covariances Q_k and R_k.

The classical *input estimation* (IE) [2] assumes that the unknown control input is constant, i.e. if the maneuver has started at time k, then

$$u_j = \begin{cases} 0 & \text{for } j = 0, 1, \ldots, k-1 \\ u & \text{for } j = k, \ldots, k+N-1, \end{cases} \tag{3}$$

where N denotes the detection window length. This assumption allows to use the least squares (LS) method for parameter estimation to obtain an estimate of the input u over the interval $[k, k+N)$, based on the information contained in the innovations of the Kalman filter assuming *zero-input* in the interval $[0, k+N)$.

In order to relax this restrictive and unrealistic assumption it was suggested [3] to represent the unknown input u_k as a linear combination of time functions, viz.

$$u_k = \sum_{l=1}^{p} a_l b_l(t_k), \tag{4}$$

where $b_l(t_k)$ are *known* scalar functions of time and a_l are *unknown constant* vector coefficients. For the so defined "non-constant" input the LS estimation technique has been applied and a thorough algorithm derivation has been performed. [3]

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Apparently, if we consider the input transition matrix G in (1) as time invariant then the presentation of the input (4) is more general than that of the constant input $u_k = u$ in (3). On the other hand, however, if we consider the *overall* unknown input $a \triangleq (a_1, a_2, \ldots, a_p)$, it is in fact constant and the known time functions $b_l(t_k), l = 1, \ldots, p$ influence this input as transition coefficients (in the same manner as the input transition matrix G does). That is why it is more natural that these coefficients be attributed to the input transition (coefficient) matrix rather than to the input itself. This underlying reason has led us to the following two observations.

- The generalized IE model (1) with (4) is a particular case of the *constant input* model
$$x_{k+1} = Fx_k + G_k u + w_k \tag{5}$$
with *time-varying input transition matrix* G_k. Indeed, if we set
$$a \triangleq \left[a_1' \ldots a_p'\right]' \text{ and } G_k^a \triangleq [b_1(t_k)G \mid b_2(t_k)G \mid \ldots \mid b_p(t_k)G] \tag{6}$$
then (1) with (4) can be recast as
$$x_{k+1} = F_k x_k + G_k^a a + w_k. \tag{7}$$
That is, a stands for the unknown *constant* input and G_k^a for the *known* (time-varying) input transition matrix. Of course (5) comprises (7) and is not restricted to the particular choice of G_k as G_k^a.

- The classical IE for constant input is *valid* for time-varying systems, and in particular for (5) (respectively (7)).

These observations imply that the GIE of [3] is a particular case of the classical IE with time-varying input transition matrix G_k (if we set $G_k = G_k^a$) . Next, we describe in more details the IE for time-varying systems and show how the GIE can be obtained from it.

2. IE for Time-Varying Systems

Although the IE method of Chan, Hu and Plant,[2] has been traditionally treated in time invariant system setting, it is valid for time-varying systems as well and no additional difficulties appear in this consideration. We summarize the basic IE method with reference to the target model (5).

The optimal Kalman filter (KF) for the system (5), (2), where F_k may be also time-varying, is [4]

$$\begin{aligned}
\hat{x}_{k+1} &= (I - K_{k+1}H)\,F_k\hat{x}_k + (I - K_{k+1}H)\,G_k u_k + K_{k+1}z_{k+1} && (8)\\
P_{k+1} &= (I - K_{k+1}H)\,(F_k P_k F_k' + Q_k) && (9)\\
K_{k+1} &= (F_k P_k F_k' + Q_k)\,H'\left[H\,(F_k P_k F_k' + Q_k)\,H' + R_{k+1}\right]^{-1}. && (10)
\end{aligned}$$

Let the assumption (3) holds and denote by $\hat{x}_j^*$ and $\hat{x}_j$ the estimates of the *hypothetical* Kalman filter with the *correct input* u_j of (3), and the *real* KF, running with the *zero-input model* $u_j = 0,\ j = k, \ldots, k+N-1$, respectively. Their residuals, defined respectively as

$$\tilde{z}_j^* \triangleq z_j - H\hat{x}_j^*, \quad \tilde{z}_j \triangleq z_j - H\hat{x}_j, \quad j = 1, 2, \ldots \tag{11}$$

satisfy

$$\tilde{z}_{k+n} = HD_{k+n}u + \tilde{z}_{k+n}^*, \quad n = 1, 2, \ldots, N\ , \tag{12}$$

where

$$D_{k+n} \triangleq \sum_{i=1}^{n} \prod_{j=n}^{i+1} (I - K_{k+j}H)\,F_{k+j}\,(I - K_{k+i}H)\,G_{k+i-1}. \tag{13}$$

It is known that $\left\{\tilde{z}_{k+n}^*\right\}_{n=1,2,\ldots,N}$ is a white noise sequence with $\tilde{z}_{k+n}^* \sim \mathcal{N}\,(0, S_{k+n})$, where the covariance $S_{k+n} = HP_{k+n}H' + R_{k+n}$.[4] Thus according to (12) the residuals of the real (zero-input) KF provide noisy measurements of the unknown input, and the *best linear unbiased estimate* (BLUE) of u can be straightforwardly obtained by means of the LS method for this system.[4]

Specifically, the system (12) can be recast in the "batch form"

$$\tilde{\mathcal{Z}} = \mathcal{H}u + \tilde{\mathcal{Z}}^*, \tag{14}$$

where stacked vectors and matrices are denoted as follows

$$\begin{aligned}
\tilde{\mathcal{Z}} &= \begin{bmatrix} \tilde{z}_{k+1}' & \ldots & \tilde{z}_{k+N}' \end{bmatrix}',\\
\mathcal{H} &= \begin{bmatrix} (HD_{k+1})' & \ldots & (HD_{k+N})' \end{bmatrix}',\\
\tilde{\mathcal{Z}}^* &= \begin{bmatrix} \tilde{z}_{k+1}^{*\prime} & \ldots & \tilde{z}_{k+N}^{*\prime} \end{bmatrix}'
\end{aligned} \tag{15}$$

and $\tilde{\mathcal{Z}}^* \sim \mathcal{N}\,(0, \mathcal{S})$ for $\mathcal{S}$ =block-diag$\left\{\ S_{k+1},\ \ldots,\ S_{k+N}\ \right\}$. Then the BLUE of u which minimizes the normalized error

$$\mathcal{L}_{\mathrm{LS}}(u) \triangleq \tilde{\mathcal{Z}}^{*\prime}\mathcal{S}^{-1}\tilde{\mathcal{Z}}^* = \left(\tilde{\mathcal{Z}} - \mathcal{H}u\right)'\mathcal{S}^{-1}\left(\tilde{\mathcal{Z}} - \mathcal{H}u\right) \tag{16}$$

is

$$\hat{u} = \mathcal{P}\mathcal{H}'\mathcal{S}^{-1}\tilde{\mathcal{Z}} \text{ with covariance } \mathcal{P} = \left(\mathcal{H}'\mathcal{S}^{-1}\mathcal{H}\right)^{-1} \tag{17}$$

The minimal normalized error is given by

$$\hat{\mathcal{L}}_{\text{LS}} \triangleq \mathcal{L}_{\text{LS}}(\hat{u}) = \left(\tilde{\mathcal{Z}} - \mathcal{H}\hat{u}\right)' \mathcal{S}^{-1} \left(\tilde{\mathcal{Z}} - \mathcal{H}\hat{u}\right) = \tilde{\mathcal{Z}}'\mathcal{S}^{-1}\tilde{\mathcal{Z}} - \Delta\mathcal{L}_{\text{LS}}(\hat{u}) \tag{18}$$

with

$$\Delta\mathcal{L}_{\text{LS}}(\hat{u}) = \hat{u}'\mathcal{P}^{-1}\hat{u} = \left(\mathcal{H}'\mathcal{S}^{-1}\tilde{\mathcal{Z}}\right)' \mathcal{P} \left(\mathcal{H}'\mathcal{S}^{-1}\tilde{\mathcal{Z}}\right) \tag{19}$$

and $\Delta\mathcal{L}_{\text{LS}}(\hat{u})$ is $\chi^2_{n_u}$ distributed, provided the true input u is zero.[1]

Thus, the first stage of the common IE method – *estimation of the input* is performed via (17). The second stage – *maneuver detection* – realizes the *significance test* [4]

$$\Delta\mathcal{L}_{\text{LS}} > \lambda \tag{20}$$

through (19), where λ is chosen for a given P_{FA}. The third stage of the IE algorithm – *estimate correction* – is performed in case of detecting a maneuver according to

$$\hat{x}^u_{k+N} = \hat{x}_{k+N} + \underbrace{D_{k+N}\hat{u}}_{\text{correction term}} \tag{21}$$

$$P^u_{k+N} = P_{k+N} + \underbrace{D_{k+N}\mathcal{P}D'_{k+N}}_{\text{uncertainty increase}} \tag{22}$$

In the above, we very briefly recalled the known IE method with the sole difference of considering the *generic time-varying* target model.

3. GIE as a Corollary of IE

Now that the IE is available one can obtain the GIE algorithm of [3] (specifically, the results presented in sections III and IV therein) as a corollary of the above given common IE. Although it should be apparent from the two remarks made in the Introduction we illustrate some details.

Consider the problem as formulated by Lee and Tahk . [3] Let us set for this problem a and G^a_k as in (6) and substitute G_k with G^a_k and u with a throughout in the Eqns (13) – (22) of the IE.

After some routine formulae manipulations the following key intermediate relations can be subsequently obtained

$$D_{k+n} =$$
$$\sum_{i=1}^{n} M_{k+n}^{k+i} \left(I - K_{k+i} H \right) G_{k+i-1} = \tag{23}$$
$$\sum_{i=1}^{n} M_{k+n}^{k+i} \left(I - K_{k+i} H \right) \left[b_1(t_{k+i-1}) G \mid \ldots \mid b_p(t_{k+i-1}) G \right] = \tag{24}$$
$$\left[\sum_{i=1}^{n} M_{k+n}^{k+i} \left(I - K_{k+i} H \right) N_{k+i} b_1(t_{k+i-1}) \mid \ldots \mid \sum_{i=1}^{n} M_{k+n}^{k+i} \left(I - K_{k+i} H \right) N_{k+i} b_p(t_{k+i-1}) \right] = \tag{25}$$
$$\left[D_{k+n}^{1} \mid \ldots \mid D_{k+n}^{p} \right] \tag{26}$$

$$HD_{k+n} = \left[C_{k+n}^{1} \mid \ldots \mid C_{k+n}^{p} \right] \text{ since } HD_{k+n}^{l} = C_{k+n}^{l}, \quad l = 1, 2, \ldots, p \tag{27}$$
$$HD_{k+n} u = \sum_{i=1}^{n} A_{k+n}^{k+i} \sum_{l=1}^{p} a_l b_l(t_{k+i-1}) \text{ since } G_{k+i-1} u \triangleq G_{k+i-1}^{a} a = G \sum_{l=1}^{p} a_l b_l(t_{k+i-1}) \tag{28}$$
$$\mathcal{H}' \mathcal{S}^{-1} \tilde{\mathcal{Z}} = \sum_{n=1}^{N} \begin{bmatrix} (C_{k+n}^{1})' \\ \vdots \\ (C_{k+n}^{p})' \end{bmatrix} S_{k+n}^{-1} \tilde{z}_{k+n} \triangleq \Omega \tag{29}$$
$$\mathcal{H}' \mathcal{S}^{-1} \mathcal{H} = \sum_{n=1}^{N} \begin{bmatrix} (C_{k+n}^{1})' \\ \vdots \\ (C_{k+n}^{p})' \end{bmatrix} S_{k+n}^{-1} \left[C_{k+n}^{1} \mid \ldots \mid C_{k+n}^{p} \right] \triangleq \Gamma, \tag{30}$$

where all quantities M_{k+n}^{k+i}, N_{k+i}, D_{k+n}^{l}, C_{k+n}^{l}, A_{k+n}^{k+i}, Ω, Γ are the same as defined by Lee and Tahk.[3]

Consider now the IE algorithm. Firstly, the error (16) is

$$\mathcal{L}_{\mathrm{LS}}(u) = \sum_{n=1}^{N} \left(\tilde{z}_{k+n} - HD_{k+n} u \right)' S_{k+n}^{-1} \left(\tilde{z}_{k+n} - HD_{k+n} u \right) \tag{31}$$

and after the substitution of $G_k u$ with $G_k^a a$, in view of (28), it transforms to the performance index $L(k, N)$ defined in[3] through the identity $-\frac{1}{2}\mathcal{L}_{\mathrm{LS}}(u) = L(k, N)$ (as in Eqn. (12) of [3]). Secondly, the IE equation (17) after the substitutions (29), (30) leads to the basic GIE equation (20) of [3], since

$$\mathcal{H}' \mathcal{S}^{-1} \tilde{\mathcal{Z}} = \Omega \text{ and } \mathcal{P}^{-1} \triangleq \mathcal{H}' \mathcal{S}^{-1} \mathcal{H} = \Gamma. \tag{32}$$

Further, in view of (32), Lema 1, Lema 2, and the maneuver detector ((28) of [3]) immediately follow from (18), (19), and (20) respectively. Finally, (21) and (22) yield the correction equations (30) and (31) of [3], respectively, that can be seen by accounting for (24).

Thus we proved that the GIE algorithm of [3] can be obtained from the common IE algorithm applied to the particular choice of G_k as G_k^a, and u as a.

4. Conclusion

More insight has been given to the problem of input estimation. It has been shown that the so called generalized input estimation can be interpreted as a particular case of the conventional input estimation with *constant input* and *time-varying transition matrix of the input*. The latter setting, however, is more general than that of the generalized input estimation. In practice, it enables designing models with various "non-constant" inputs to be done through the design of the input transition matrix.

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Notes:

[1] Xiao Rong Li and Vesselin Jilkov, "A Survey of Maneuvering Target Tracking—Part IV: Decision-Based Methods," in *Proc. SPIE Conf. Signal and Data Processing of Small Targets,* vol. 4728 (Orlando, Florida: April 2002).

[2] Y. T. Chan, A. G. C. Hu, and J. B. Plant, "A Kalman Filter Based Tracking Scheme with Input Estimation," *IEEE Trans. AES*, AES-15, 2, (Mar. 1979): 237–244.

[3] H. Lee and M.-J. Tahk, "Generalized input-estimation technique for tracking maneuvering targets," *IEEE Transactions on Aerospace and Electronic Systems* 35, 4 (1999): 1388–1402.

[4] Yaakov Bar-Shalom, Xiao Rong Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation: Theory, Algorithms, and Software* (New York: Wiley, 2001).

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